Signal Analysis & Communication ECE 355

Ch. 3.5: Properties of Continuous Time Fourier Series

Lecture 17

18 - 10 - 2023

Ch.3·5: PROPERTIES OF CONTINUOUS TIME FOURIER SERIES (CTFS)

- Suppose we have the CTFS coefficients $\{a_k\}_{k=-\infty-\infty}$ of some sig. $x(t)$.
- How can we find the CTFS coefficients of signals obtained by simple manipulations, e.g., $3x(t)+2$, $x(t-5)$, $Re\{x(t)\}$, $\frac{dx(t)}{dt}$ ...?
- There are many useful properties of CTFS that you can use as shortcuts to find the CTFS coefficients for these transformed signals.
- First, for convenience, let's use a shorthand notation to indicate the relationship between a periodic sig. & its CTFS coefficients.

$$\boxed{x(t) \xleftrightarrow{FS} a_k}$$

$x(t)$ - CT Periodic

Fundamental Period - T

Fundamental Frequency $\omega_0 = \frac{2\pi}{T}$

① Linearity:

If $y(t)$ has a fundamental period T (same as of $x(t)$)
then

$$\boxed{Ax(t)+By(t) \xleftrightarrow{FS} Aa_k + Bb_k}$$

→ same ω (as same T)

Proof:

$$LHS = A\left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right) + B\left(\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}\right)$$

$$= \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k) e^{jk\omega_0 t}$$

② Time Shifting:

$$\boxed{x_{(t-t_0)} \overset{FS}{\longleftrightarrow} \underline{a_k}\, e^{-jk\omega_0 t_0}}$$

No change to the magnitude of the FS coefficients.

<u>Proof:</u>

Finding FS coefficients for $x_{(t-t_0)}$

$$= \frac{1}{T}\int_T x_{(t-t_0)}\, e^{-jk\omega_0 t}\, dt$$

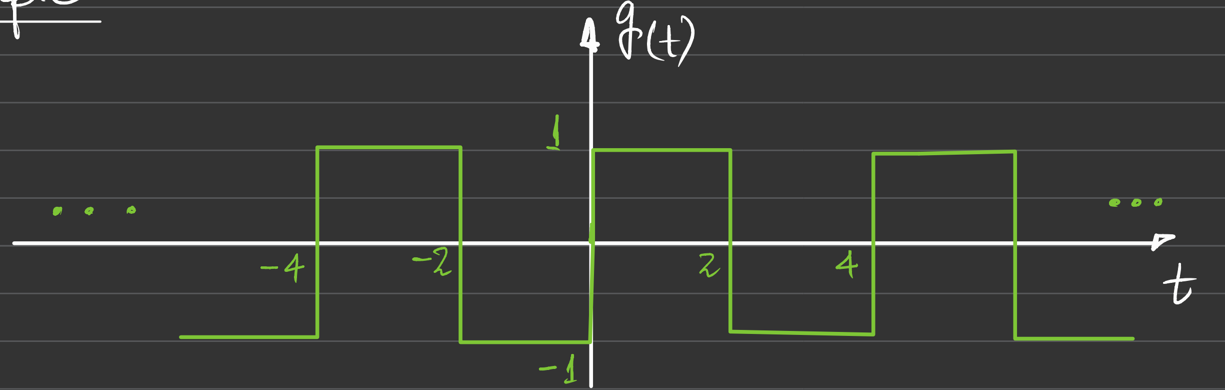
$$\overset{\tau = t - t_0}{=} \frac{1}{T}\int_T x_{(\tau)}\, e^{-jk\omega_0(\tau + t_0)}\, d\tau$$

unchanged as it is over any period!

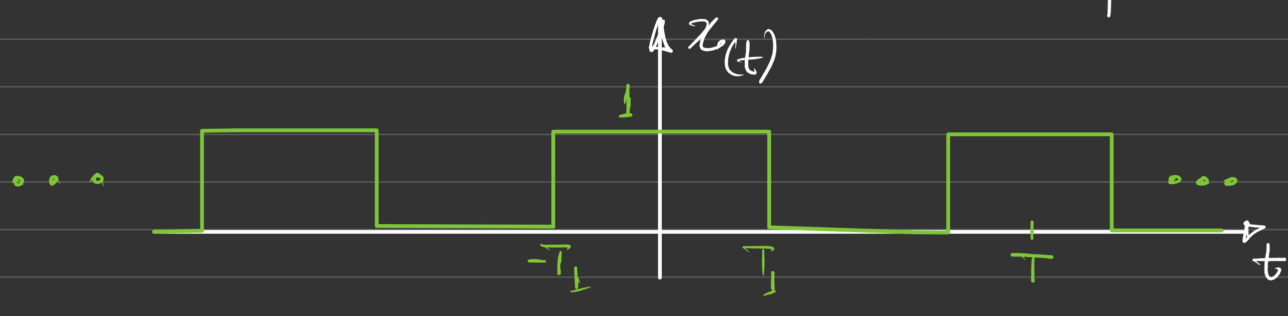
$$= e^{-jk\omega_0 t_0}\, \underbrace{\frac{1}{T}\int_T x_{(\tau)}\, e^{-jk\omega_0 \tau}\, d\tau}_{a_k}$$

$$= a_k\, e^{-jk\omega_0 t_0}$$

<u>Example</u>



- To evaluate its FS coefficients, recall the example of Lec. 15.



$$\Big\downarrow FS$$

$$a_k = \begin{cases} \dfrac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} & , \quad k \neq 0 \\ 1/2 & , \quad k = 0 \end{cases}$$

- Relating $g(t)$ to $x(t)$: Let $T=4$, $\omega_0 = 2\pi/4 = \pi/2$

$$\& \quad g(t) = \underbrace{2x(t-1)}_{\mathrm{I}} \underbrace{-1}_{\mathrm{II}} \longleftrightarrow c_k \quad — ①$$

$$\mathrm{I.} \quad 2x(t-1) \xleftrightarrow{FS} 2a_k e^{-jk\pi/2(1)}$$

$$\mathrm{II.} \quad -1 \xleftrightarrow{FS} \begin{cases} -1 & , \quad k=0 \\ 0 & , \quad k \neq 0 \end{cases}$$

using FS analysis Eqn.

$$\therefore \mathrm{I} \,\&\, \mathrm{II} \Rightarrow \begin{cases} c_0 = 2a_0 - 1 = 1 - 1 = 0 & ; \quad k=0 \\ c_k = 2e^{-jk\pi/2} \dfrac{\sin k\pi/2}{k\pi} & ; \quad k \neq 0 \end{cases}$$

③ <u>Time Scaling:</u>

- For $\alpha > 0$, $x(\alpha t) \xleftrightarrow{FS} a_k$, with fundamental freq. $\alpha\omega_0$.

<u>Proof:</u>

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(\alpha t)}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

Fundamental freq. now is '$\alpha\omega_0$'

④ Time Reversal:

$$x_{(-t)} \xleftrightarrow{FS} a_{-k}$$

Proof:

$$x_{(-t)} = \sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0(-t)}$$

$$x_{(-t)} \overset{m=-k}{=} \sum_{k=-\infty}^{\infty} \underbrace{a_{-k}}_{\text{FS coefficient of } x_{(-t)}} e^{jk\omega_0 t}$$

SPECIAL CASES

I. $x_{(t)}$ – Even

$$\Rightarrow x_{(t)} = x_{(-t)}$$

$$a_k = a_{-k}$$

$\Rightarrow$ Even a_k

II. $x_{(t)}$ – Odd

$$\Rightarrow x_{(t)} = -x_{(-t)}$$

$$a_k = -a_{-k}$$

$\Rightarrow$ Odd a_k

⑤ Conjugation:

$$x^*_{(t)} \xleftrightarrow{FS} a^*_{-k}$$

$x^*_{(t)}$ is conjugate of $x_{(t)}$

Proof:

$$x^*_{(t)} = \left(\sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0 t}\right)^*$$

$$= \sum_{m=-\infty}^{\infty} a_m^* e^{-jm\omega_0 t}$$

$$\overset{m=-k}{=} \sum_{k=-\infty}^{\infty} \underbrace{a^*_{-k}}_{\text{FS coefficients of } x^*_{(t)}} e^{jk\omega_0 t}$$

SPECIAL CASES

I. $x_{(t)}$ - Real

$$\Rightarrow x_{(t)} = x^*(t)$$

$$\updownarrow \qquad \updownarrow$$

$$a_k = a^*_{-k}$$

$$\Rightarrow a_{-k} = a_k^*$$

"Conjugate Symmetric" (Hermitian)

(mag. is an even f^n of k whereas phase is an odd f^n of k)

- FS for this case:

$$x_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty}\left(a_k e^{jk\omega_0 t} + \underbrace{a_{-k} e^{-jk\omega_0 t}}_{=\left(a_k e^{jk\omega_0 t}\right)^*}\right)$$

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\{a_k e^{jk\omega_0 t}\}$$

$\mathrm{Re}\{z\} = \frac{z+z^*}{2}$

- If $a_k = A_k e^{j\theta_k}$

$$x_{(t)} = a_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\{A_k e^{j(k\omega_0 t + \theta_k)}\}$$

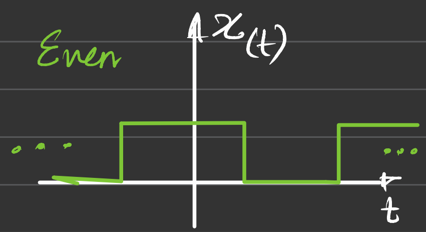
$$= a_0 + \sum_{k=1}^{\infty} 2A_k \cos(k\omega_0 t + \theta_k)$$

II. $x_{(t)}$ — Real & Even

$$\Rightarrow a_k = a_{-k} = a_k^*$$

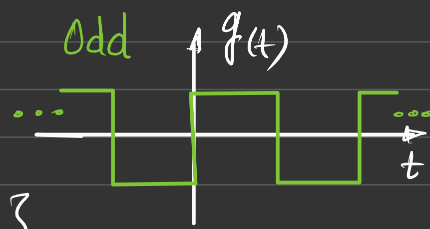
$$\Rightarrow \text{Real \& Even } \{a_k\}$$

Even



III. $x_{(t)}$ — Real & Odd

Odd



$$\Rightarrow a_k = -a_{-k} = -a_k^*$$

$$\Rightarrow \text{Purely imaginary \& odd } \{a_k\}$$

⑥ <u>Multiplication:</u>

If $\quad x_{(t)} \xleftrightarrow{FS} a_k \quad \& \quad y_{(t)} \xleftrightarrow{FS} b_k$

then $\quad x_{(t)} y_{(t)} \xleftrightarrow{FS} c_k$

where $\quad c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$ $\quad$ Convolution in DT

<u>Proof:</u>

$$x_{(t)} y_{(t)} = \left(\sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0 t}\right)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_l b_m e^{j(l+m)\omega_0 t}$$

$$\overset{k=l+m}{=} \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} a_l b_{k-l}\right) e^{jk\omega_0 t}$$

⑦ Parseval's Relation:

- If $x_{(t)} \xleftrightarrow{FS} a_k$

then
$$\boxed{\frac{1}{T}\int_T |x_{(t)}|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2}$$

Parseval's Power Theorem

Proof:

From conjugation property ⑤

$$x^*_{(t)} \longleftrightarrow a^*_{-k}$$

From multiplication property ⑥

$$\underbrace{x_{(t)}\, x^*_{(t)}}_{=|x_{(t)}|^2} \longleftrightarrow c_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}$$

$$= \sum_{l=-\infty}^{\infty} a_l\, a^*_{l-k}$$

$$\text{Average of } |x_{(t)}|^2 = \frac{1}{T}\int_T |x_{(t)}|^2 dt$$

$$c_0 = \sum_{l=-\infty}^{\infty} a_l\, a_l^*$$

$$= \sum_{l=-\infty}^{\infty} |a_l|^2$$

NOTE: Average Power of the kth component $= \frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 dt$

$$= \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2$$

OTHER CTFS PROPERTIES: Table 3.1 (textbook)